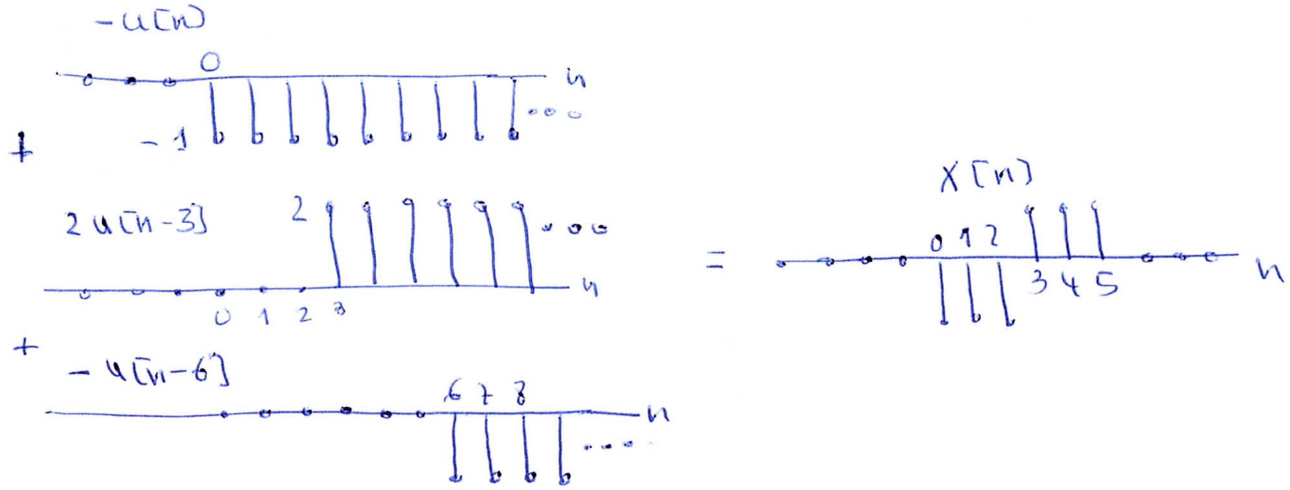
PROBLEMA 1: Calcular la convolución de las siguientes señales:

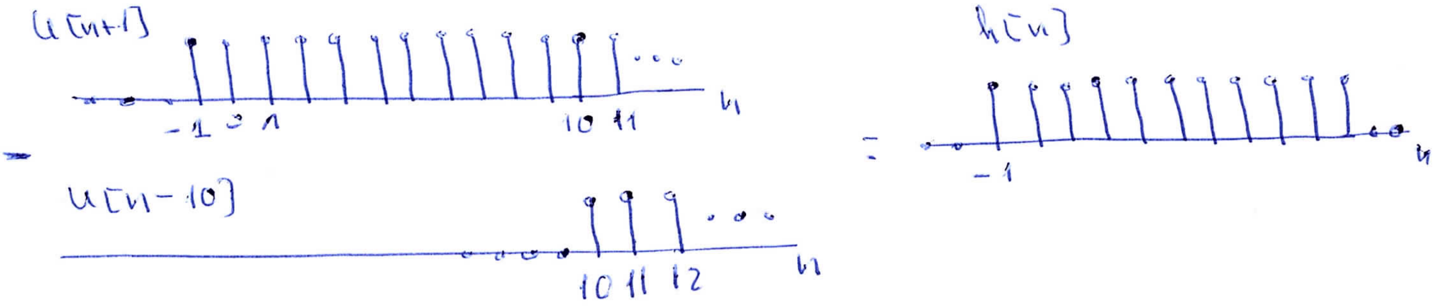
a) $x[n] = -u[n] + 2u[n-3] - u[n-6]$

$h[n] = u[n+1] - u[n-10]$

Representamos $x[n]$:



Ahora dibujamos $h[n]$



es decir,

$$x[n] = \begin{cases} 0, & n < 0 \\ -1, & 0 \leq n \leq 2 \\ 1, & 3 \leq n \leq 5 \\ 0 & n > 5 \end{cases} \qquad h[n] = \begin{cases} 1, & -1 \leq n \leq 9 \\ 0 & \text{otro } n \end{cases}$$

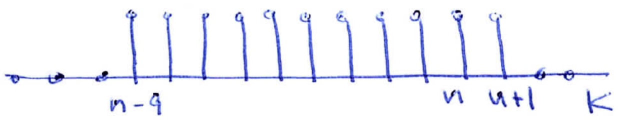
Calculamos la convolución de las dos señales discretas:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

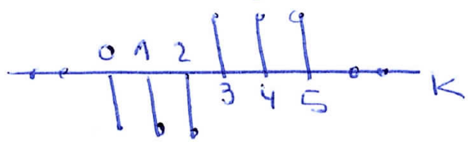
Escribimos y representamos $x[k]$ y $h[n-k]$:

$$x[k] = \begin{cases} 0, & k < 0 \\ -1, & 0 \le k \le 2 \\ 1, & 3 \le k \le 5 \\ 0, & k > 5 \end{cases} \quad ; \quad h[n-k] = \begin{cases} 1, & n-9 \le k \le n+1 \\ 0, & \text{otro } n \end{cases}$$

$h[n-k]$



$x[k]$



Analizamos los distintos intervalos:

I1) $n+1 < 0$: $n < -1$, $y[n] = 0$

I2) $\left.\begin{matrix} n+1 \ge 0 \\ n+1 \le 2 \end{matrix}\right\} -1 \le n \le 1$, $y[n] = \sum_{k=0}^{n+1} (-1)\cdot 1 = -(n+2)$

I3) $\left.\begin{matrix} n+1 > 2 \\ n+1 \le 5 \end{matrix}\right\} 1 < n \le 4$, $y[n] = \sum_{k=0}^{2} (-1)\cdot 1 + \sum_{k=3}^{n+1} 1\cdot 1 =$

$= 3\cdot(-1) + \sum_{k=3}^{n+1} 1$

c.v: $\left| \begin{matrix} l = k-3 \\ k=3 \to l=0 \\ k=n+1 = n-2 \end{matrix} \right. \Rightarrow \sum_{k=3}^{n+1} 1 = \sum_{l=0}^{n-2} 1 = (n-1)\cdot 1$

luego $y[n] = 3(-1) + (n-1)\cdot 1 = -3 + (n-1) = n-4$

I4) $\left.\begin{matrix} n+1 > 5 \\ n-9 \le 0 \end{matrix}\right\} 4 < n \le 9$, $y[n] = \sum_{k=0}^{2} (-1)\cdot 1 + \sum_{k=3}^{5} 1\cdot 1 =$

$= 3\cdot(-1) + 3\cdot 1 = 0$

I5) $\left.\begin{matrix} n-9 > 0 \\ n-9 \le 2 \end{matrix}\right\} 9 < n \le 11$, $y[n] = \sum_{k=n-9}^{2} (-1)\cdot 1 + \sum_{k=3}^{5} 1\cdot 1 =$

$$\text{c.v}: \left\{ \begin{array}{l} l = k-(n-9) \\ k = n-9,\ l=0 \\ k=2,\ l = 11-n \end{array} \right\} \quad y[n] = \sum_{k=0}^{11-n} (-1)\cdot 1 + \sum_{k=3}^{5} 1\cdot 1 =$$

$$= (12-n)(-1) + 3\cdot 1 = n-9$$

I6) $\left. \begin{array}{l} n-9>2 \\ n-9 \leq 5 \end{array} \right\} \quad 11 < n \leq 14,\ y[n] = \sum_{k=n-9}^{5} 1\cdot 1 = \sum_{l=0}^{14-n} 1 = 15-n$

I7) $n-9>5:\ n>14,\ y[n]=0$

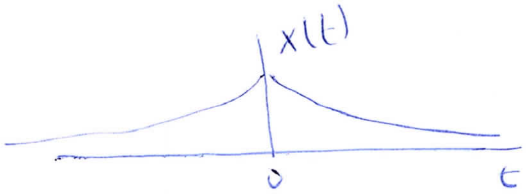
$$y[n] \left\{ \begin{array}{ll} 0, & n<-1 \\ -(n+2), & -1 \leq n \leq 1 \\ n-4, & 1<n\leq 4 \\ 0, & -4<n\leq 9 \\ n-9, & 9<n\leq 11 \\ 15-n, & 11<n\leq 14 \\ 0, & n>14 \end{array} \right.$$

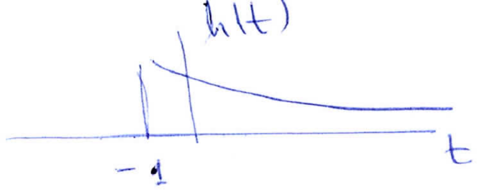
b) $x(t) = e^{-|t|}$

$h(t) = e^{-2(t+1)} u(t+1)$

Escribimos $x(t)$ y $h(t)$ como:

$$x(t) = \left\{ \begin{array}{ll} e^{-t}, & t>0 \\ e^{t}, & t<0 \end{array} \right. ; \quad h(t) = \left\{ \begin{array}{ll} e^{-2(t+1)}, & t>-1 \\ 0, & \text{otro } t \end{array} \right.$$





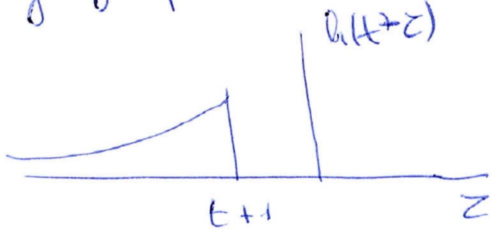
Como las señales son continuas, calculamos la integral de convolución:

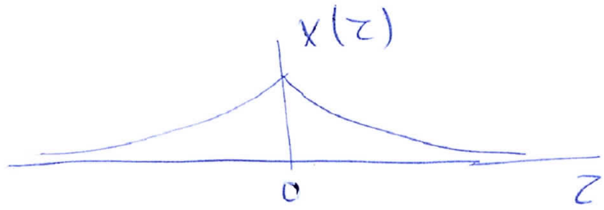
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

En función de la variable τ:

$$x(\tau) = \begin{cases} e^{-\tau}, & \tau > 0 \\ e^{\tau}, & \tau < 0 \end{cases} \quad ; \quad h(t-\tau) = \begin{cases} e^{-2(t-\tau+1)}, & \tau < t+1 \\ 0, & \text{otro } \tau \end{cases}$$

y gráficamente:





Analizamos los distintos intervalos:

I1) $t+1 < 0$: $t < -1$, $y(t) = \int_{-\infty}^{t+1} e^{\tau} \cdot e^{-2(t-\tau+1)}\, d\tau =$

$$= e^{-2(t+1)} \int_{-\infty}^{t+1} \underbrace{e^{\tau} \cdot e^{2\tau}}_{e^{3\tau}}\, d\tau = \frac{1}{3} e^{-2(t+1)} e^{3\tau}\Big|_{-\infty}^{t+1} =$$

$$= \frac{1}{3} e^{-2(t+1)} \cdot e^{3(t+1)} = \frac{1}{3} e^{(t+1)}$$

I2) $t+1 > 0$: $t > -1$, $y(t) = \int_{-\infty}^{0} e^{\tau} \cdot e^{-2(t-\tau+1)}\, d\tau + \int_{0}^{t+1} e^{-\tau} \cdot e^{-2(t-\tau+1)}\, d\tau$

$$= e^{-2(t+1)} \int_{-\infty}^{0} e^{3\tau}\, d\tau + e^{-2(t+1)} \int_{0}^{t+1} e^{\tau}\, d\tau =$$

$$= e^{-2(t+1)} \cdot \frac{1}{3} e^{3\tau}\Big|_{-\infty}^{0} + e^{-2(t+1)} \cdot e^{\tau}\Big|_{0}^{t+1} =$$

$$= \frac{1}{3} e^{-2(t+1)} + e^{-2(t+1)} \left(e^{t+1} - 1\right) = e^{-(t+1)} - \frac{2}{3} e^{-2(t+1)}$$

$$y(t) = \begin{cases} \frac{1}{3} e^{(t+1)}, & t < -1 \\ e^{-(t+1)} - \frac{2}{3} e^{-2(t+1)}, & t > -1 \end{cases}$$

PROBLEMA 2. Determinar si los siguientes sistemas LTI son causales y estables. Razonar la respuesta.

Un sistema LTI es causal si:

$h[n] = 0, \quad n < 0$

$h(t) = 0, \quad t < 0$

Un sistema LTI es estable si:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

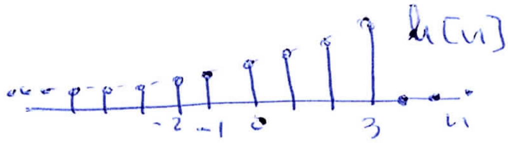
$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Analizamos las respuestas al impulso propuestas:

a) $h[n] = 2^n u[3-n]$

Analizando la función escalón podemos decir que la respuesta al impulso $h[n]$ del sistema LTI es:

$$h[n] = \begin{cases} 2^n, & n \leq 3 \\ 0, & n > 3 \end{cases}$$

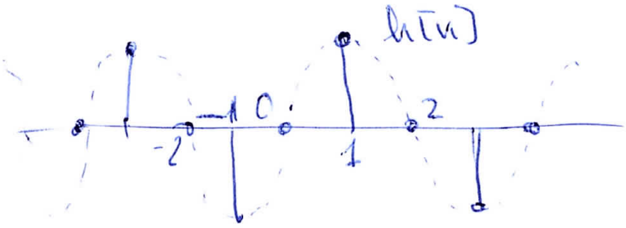


S. NO CAUSAL: ya que $h[n]$ es distinto de cero para valores negativos de n, $(n \leq 3)$

SÍ ESTABLE:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{3} 2^n = \sum_{n=-3}^{\infty} 2^{-n} = 8 + 4 + 2 + \frac{1}{1 - \frac{1}{2}} = 16$$

b) $h[n] = \sin\left(\frac{\Pi}{2} n\right)$



Sistema NO CAUSAL pues: $h[n] \neq 0$ si $n < 0$

Para comprobar si es estable:

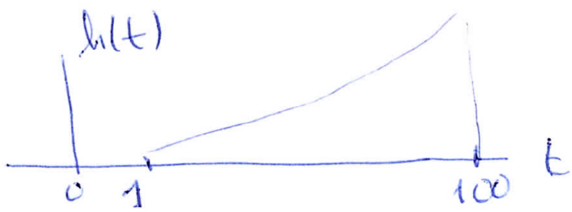
$$\sum_{n=-\infty}^{\infty} |h[n]|$$ y observamos que para los n impares $|h[n]| = 1$ luego:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \infty$$ NO ESTABLE.

c) $h(t) = e^{15t}\left(u(t-1) - u(t-100)\right)$

Podemos escribir h(t) como:

$$h(t) = \begin{cases} e^{15t}, & 1 < t < 100 \\ 0, & \text{otro } t \end{cases}$$



Como $h(t) = 0$ si $t < 1$ se puede afirmar que el sistema es CAUSAL.

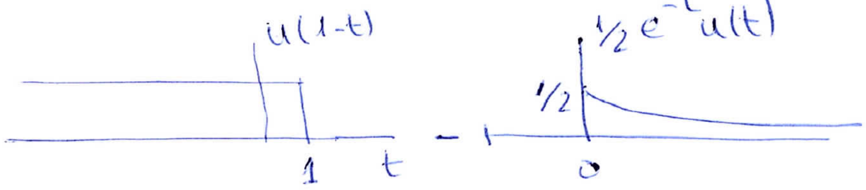
Analizamos la estabilidad:

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{1}^{100} e^{15t}\, dt = \frac{1}{15} e^{15t}\Big|_{1}^{100} =$$

$$= \frac{1}{15}\left(e^{1500} - e^{15}\right) < \infty$$

Luego el sistema es ESTABLE

d) $h(t) = u(1-t) - \frac{1}{2} e^{-t} u(t)$



El sistema es NO CAUSAL pues $h(t)$ tiene valores $\neq 0$ para t negativo.

Respecto a la estabilidad:

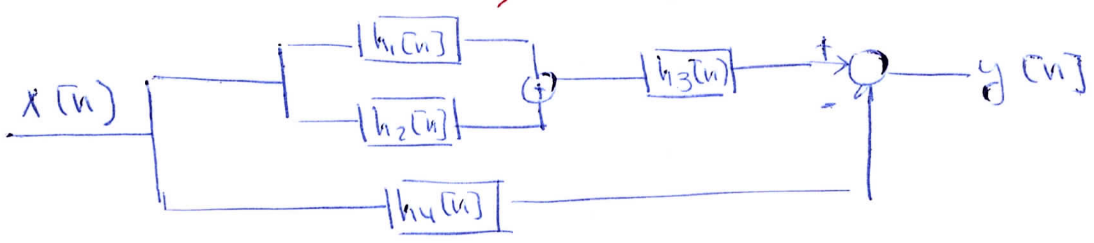
$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{1} 1 \cdot dt - \frac{1}{2} \int_{0}^{\infty} e^{-t} dt =$$

$$= t \Big|_{-\infty}^{1} + \frac{1}{2} e^{-t} \Big|_{0}^{\infty} = \infty \Rightarrow \text{NO ESTABLE}$$

PROBLEMA 3: Calcular la respuesta al impulso del sistema equivalente al mostrado en la figura. Las respuestas al impulso de cada sistema vienen dadas por:

$h_1[n] = u[n]$; $\quad h_3[n] = \delta[n-2]$

$h_2[n] = u[n+2] - u[n]$; $\quad h_4[n] = \alpha^n u[n]$



El sistema equivalente tendrá por respuesta al impulso:

$$h[n] = \left[(h_1[n] + h_2[n]) * h_3[n]\right] - h_4[n]$$

$$h_{12}[n] = h_1[n] + h_2[n] = u[n] + u[n+2] - u[n] = u[n+2]$$

$$h_{12}[n] * h_3[n] = u[n+2] * \delta[n-2] = u[n+2-2] = u[n]$$

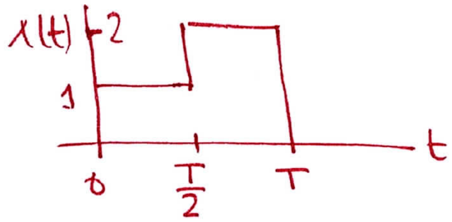
$$h[n] = h_{123}[n] - h_4[n] = u[n] - \alpha^n u[n] = u[n](1-\alpha^n) =$$

$$= \begin{cases} 1-\alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

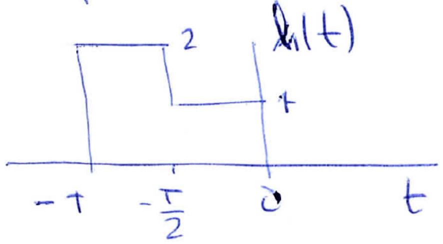
PROBLEMA 4. Calcular gráficamente la siguiente convolución:

$$y(t) = x(t) * h(t) \quad \text{si} \quad h(t) = x(-t)$$

siendo $x(t)$:

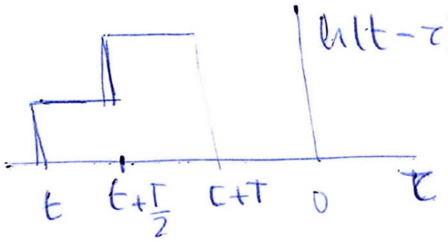


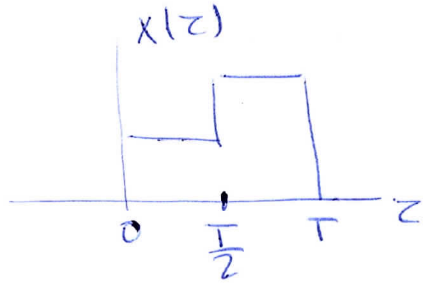
Representamos $h(t)$:



La integral de convolución es:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

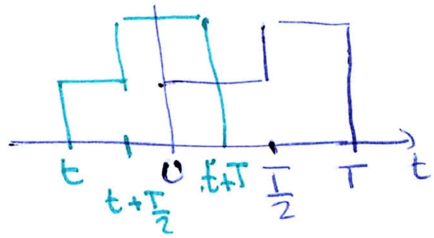




Comenzamos el cálculo de la integral de convolución:

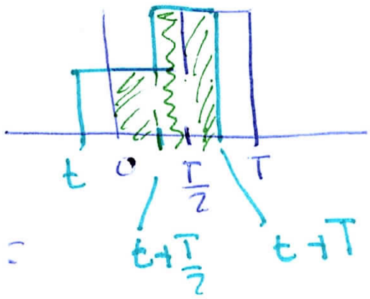
I1) $t+T<0$: $t<-T$, $y(t)=0$

I2) $\left.\begin{array}{l} t+T>0 \\ t+T<\frac{T}{2} \end{array}\right\} \left.\begin{array}{l} t>-T \\ t<-\frac{T}{2} \end{array}\right\} -T<t<-\frac{T}{2}$



$$y(t) = \int_0^{t+T} 1\cdot 2\, d\tau = 2\tau \Big|_0^{t+T} = 2(t+T)$$

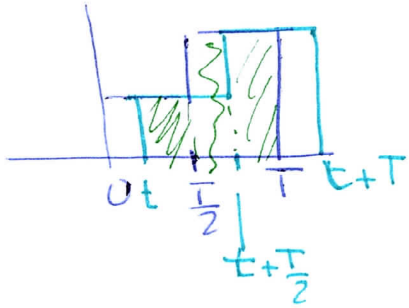
I3) $\left.\begin{array}{l} t+T > \frac{T}{2} \\ t+T < T \end{array}\right| \left.\begin{array}{l} t > -\frac{T}{2} \\ t < 0 \end{array}\right| -\frac{T}{2} < t < 0$



$$y(t) = \int_0^{t+T/2} 1\cdot 1\, d\tau + \int_{t+T/2}^{T/2} 1\cdot 2\, d\tau + \int_{T/2}^{t+T} 2\cdot 2\, d\tau =$$

$$= \tau\Big|_0^{t+T/2} + 2\tau\Big|_{t+T/2}^{T/2} + 4\tau\Big|_{T/2}^{t+T} = 3t + \frac{5}{2}T$$

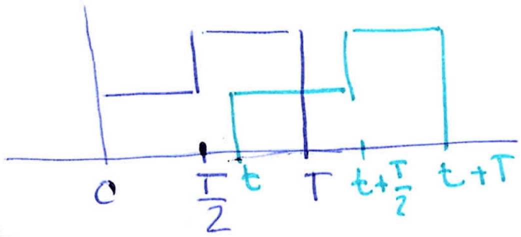
I4) $\left.\begin{array}{l} t+T > T \\ t < \frac{T}{2} \end{array}\right| \left.\begin{array}{l} t > 0 \\ t < T/2 \end{array}\right| 0 < t < \frac{T}{2}$



$$y(t) = \int_t^{T/2} 1\cdot 1\, d\tau + \int_{T/2}^{t+T/2} 2\cdot 1\, d\tau + \int_{t+\frac{T}{2}}^{T} 2\cdot 2\, d\tau =$$

$$= \tau\Big|_t^{T/2} + 2\tau\Big|_{T/2}^{t+T/2} + 4\tau\Big|_{t+T/2}^{T} = -3t + \frac{5}{2}T$$

I5) $\left.\begin{array}{l} t > \frac{T}{2} \\ t < T \end{array}\right| \frac{T}{2} < t < T$



$$y(t) = \int_t^T 2\cdot 1\, d\tau = 2\tau\Big|_t^T = 2T - 2t =$$

$$= 2(T-t)$$

I6) $t > T \quad y(t) = 0$

$$y(t) = \begin{cases} 0, & t < -T \\ 2(t+T), & -T < t < -T/2 \\ 3t + \frac{5}{2}T, & -T/2 < t < 0 \\ -3t + \frac{5}{2}T, & 0 < t < T/2 \\ 2(T-t), & T/2 < t < 1 \\ 0. & t > T \end{cases}$$

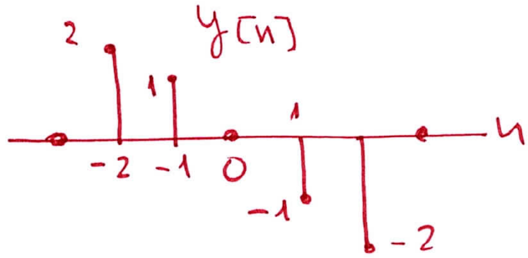
PROBLEMA 5. Considerar la conexión de sistemas representada en la figura (a). Conocida la señal de entrada $x[n]$ y la respuesta al impulso $h_2[n]$:

$$x[n] = u[n] - u[n-2]$$

$$h_2[n] = \delta[n] - \delta[n-1]$$

Calcular $h_1[n]$ si la salida $y[n]$ se corresponde con la secuencia de la figura (b):

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$



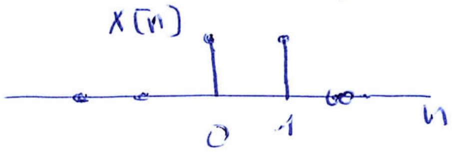
La señal de salida del sistema viene dado por:

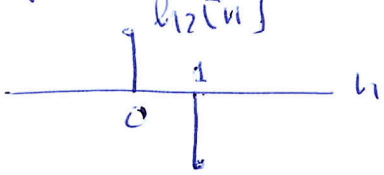
$$y[n] = x[n] * h_1[n] * h_2[n]$$

y por la propiedad conmutativa:

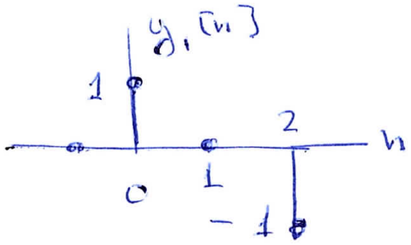
$$y[n] = x[n] * h_2[n] * h_1[n]$$

Calculamos en primer lugar $x[n] * h_2[n]$.





$$y_1[n] = x[n] * \delta[n] - x[n] * \delta[n-1] = x[n] - x[n-1]$$

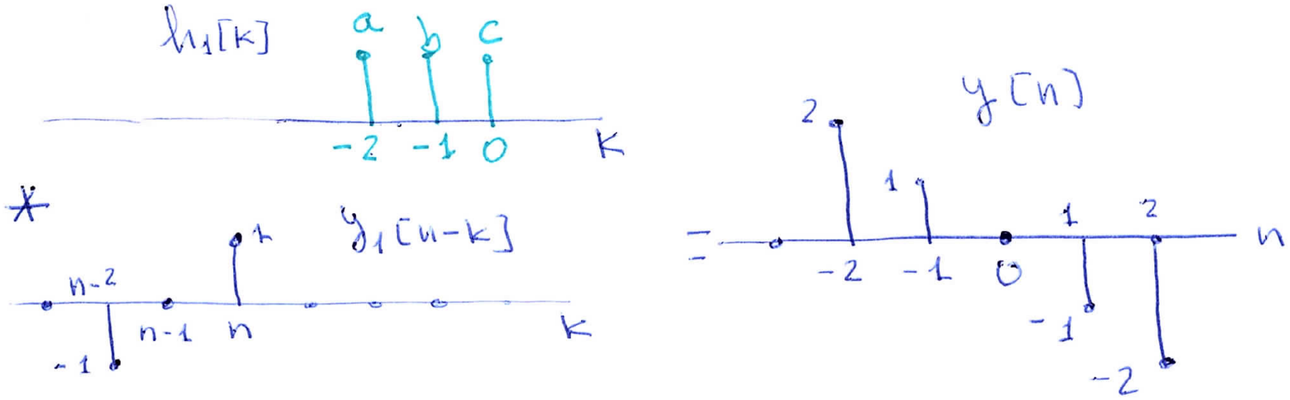


$$y_1[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 0, & n = 1 \\ -1, & n = 2 \\ 0, & n > 2 \end{cases}$$

Sabemos que $y[n] = y_1[n] * h_1[n] = h_1[n] * y_1[n]$

Planteamos la convolución gráfica:

$$y[n] = h_1[n] * y_1[n] = \sum_{k=-\infty}^{\infty} h_1[k]\, y_1[n-k]$$



Analizamos el valor inicial y final de $y[n]$:

$y[-2]$ e $y[2]$

1º solapamiento: la muestra "n" de $y_1[n-k]$ solapa a a la primera muestra de $h_1[k]$ para dar como resultado $y[-2]$

$$n = k = -2$$

y suponemos que tiene amplitud "a"

Último solapamiento: cuando la muestra "n-2" de $y_1[n-k]$ solapa la última muestra de $h_1[k]$ y ese valor será el de $y[2]$:

$$n-2 = k \Rightarrow n = k+2 = 2 \Rightarrow k = 0$$

Quedan por determinar las amplitudes a, b, c:

$n=-2$: $y[-2] = 2 = a \cdot 1 \Rightarrow a = 2$

$n=-1$ $y[-1] = 1 = b \cdot 1 + a \cdot 0 \Rightarrow b = 1$

$n=0$ $y[0] = 0 = c \cdot 1 + b \cdot 0 + a(-1) =$

$= c - a = c - 2 \Rightarrow c = 2$